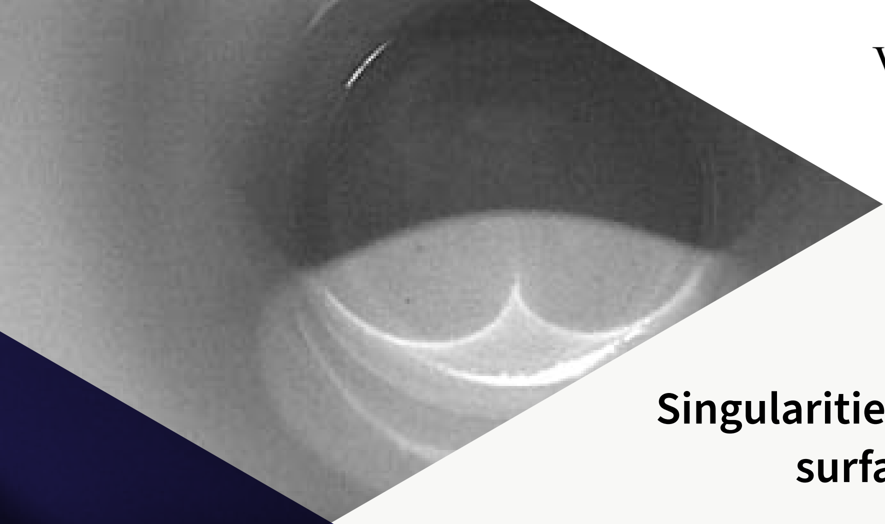

Universitat de València

Singularities of frontal surfaces

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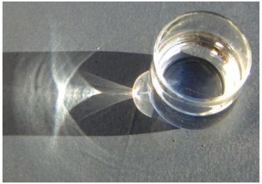
Overview

1. Introduction
2. Frontal equivalence
3. Double point space of a frontal surface

Introduction

Introduction

What is a frontal hypersurface?



(Heiner Otterstedt, 2006)
(Patrick Dirden, 2010)

- Light cardioids
- Control systems
- Sound barrier
- Hyugens’ diffraction principle

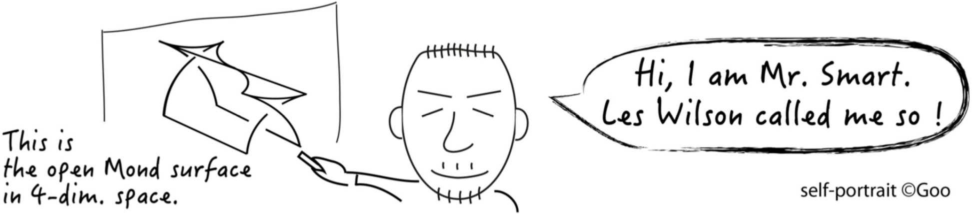
Introduction

Ishikawa's theory of infinitesimal Legendre equivalence

- Corank 1 stable frontals come from a certain family of maps, called *open Whitney umbrellas*
- Stability is codified in terms of a certain $\mathbb{C}$-algebra Q
- Legendrian codimension measures how far a frontal is from being stable

Introduction

He is Mr Smart!

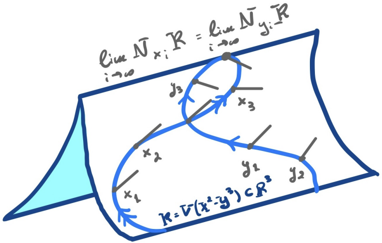


(Image by Goo Ishikawa, year unknown)

Frontal equivalence

Frontal equivalence

The flag always stays upright!



We say $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is **frontal** if there exists a $\nu\colon (\mathbb{C}^{n+1}, 0) \to T^*\mathbb{C}^{n+1}$ such that $0 \notin \nu[f(S)]$ and

$$\nu(df \circ \xi) = 0 \quad \forall \xi \in \theta_n$$

Proposition

Let $f, g\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-equivalent germs. If f is frontal, g is frontal.

Frontal equivalence
Frontal stability

- An unfolding F of f is **frontal** if it has a frontal representative. The map f is $\mathscr{F}$**-stable** if every frontal unfolding is trivial.

  Consider the vector space

$$\mathscr{F}(f) = \left\{ \left. \frac{df_t}{dt} \right|_{t=0} : (f_t, t) \text{ frontal}, f_0 = f \right\} \supseteq T\mathscr{A}_e f$$

- We say f is $\mathscr{F}$-finite if

$$\operatorname{codim}_{\mathscr{F}} f = \dim_{\mathbb{C}} \frac{\mathscr{F}(f)}{T\mathscr{A}_e f} < \infty$$

Frontal equivalence

Characterising frontal stability

Result

A corank 1 frontal surface is stable as a frontal if and only if it has frontal codimension 0.

Result

A corank 1 frontal surface f is $\mathscr{F}$-finite if and only if there is a finite representative $f\colon X \to Y$ such that $f\colon X \backslash f^{-1}(0) \to Y \backslash \{0\}$ is locally $\mathscr{F}$-stable.

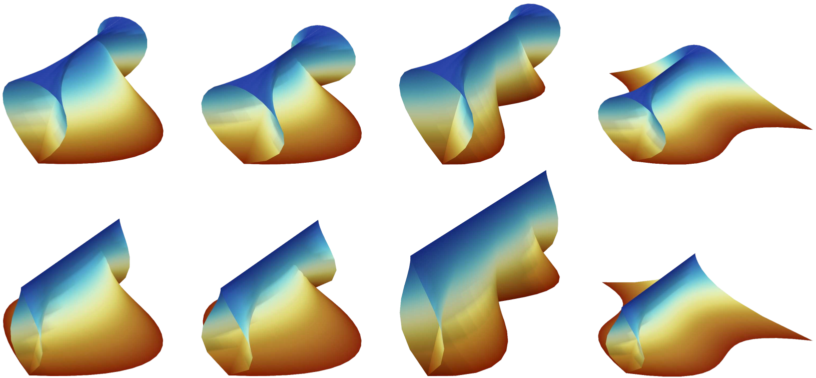
Frontal equivalence

Mond's classification of simple fold surfaces in $\mathbb{C}^3$

Mond's classification		Frontalised surface		Codimension	Notes
S_k	$y^3 + x^{k+1}y$	$\check{S}_k$	$y^5 + x^{k+1}y^3$	k	
B_k	$x^2y + y^{2k+1}$	$\check{B}_k$	$x^2y^3 + y^{2k+3}$	k	$k \geq 2$
C_k	$xy^3 + x^ky$	$\check{C}_k$	$xy^5 + x^ky^3$	k	$k \geq 3$
F_4	$y^5 + x^3y$	$\check{F}_4$	$y^7 + x^3y^3$	4	
$(x, y^2, yp(x, y^2))$		$(x, y^2, y^3p(x, y^2))$		(Mond, 1985)	$p \in \mathscr{O}_2$

Frontal equivalence

Mond's fold surfaces and their frontalisations



Frontal equivalence
Properties of frontal surfaces

- Topological invariants:
  - The only frontal fold surface that is a wave front is the cuspidal edge.
  - The $\mathscr{A}$-codimension of f is equal to the $\mathscr{F}$-codimension of $\check{f}$ (incl. ∞).
- Double point space:
  - Frontalising a fold surface introduces cuspidal edges.
  - Transverse double points are preserved.
- We can define an *extended Gaussian curvature* on frontal folds (see Saji, Umehara, and Yamada, 2009).

Frontal equivalence

Stability only depends on the branches

Given a frontal monogerm $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^{n+1})$, let

$$\tau_{\mathscr{F}}(f) = (\omega f)^{-1}[T\mathscr{K}_e f \cap \mathscr{F}(f)]|_{x=0}$$

Result

Let $S = \{x_1, ..., x_r\}$. A frontal multi-germ $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ is $\mathscr{F}$-stable as a frontal if and only if its branches,

$$f_i\colon (\mathbb{K}^n, x_i) \longrightarrow (\mathbb{K}^{n+1}, 0),$$

are $\mathscr{F}$-stable and the sets $\tau_{\mathscr{F}}(f_1), \ldots, \tau_{\mathscr{F}}(f_s)$ meet in gneneral position.

Conjecture

The set $\tau_{\mathscr{F}}(f)$ is the tangent space to the isosingular locus of f.

Frontal equivalence

Frontal reduction in corank 1

- Wavefronts can be obtained as discriminants of maps

$$F : (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{n+1}, 0)$$

  with smooth critical set (Arnol'd, 1990).
- Frontals can be generated unfolding analytic plane curves and applying a *frontal condition* to their parameters:

$$(t^3, t^4) \xrightarrow{\text{versal}} (t^3 + ut, t^4 + v_1 t + v_2 t^2) \xrightarrow{\text{frontal}} (t^3 + 2wt, t^4 + 3wt^2)$$

Frontal equivalence

Frontal reduction in corank 1

Given an unfolding $\Gamma(u,t) = (u, \gamma_u(t))$ of an analytic plane curve $\gamma\colon (\mathbb{K},0) \to (\mathbb{K}^2,0)$, the pull-back of Γ by $h\colon (\mathbb{K}^r,0) \to (\mathbb{K}^s,0)$ is defined as

$$(h^*\Gamma)(v,t) = (v, \gamma_{h(v)}(t))$$

Result

If Γ is miniversal, there exists a graph-type immersion h such that $h^*\Gamma$ is miniversal as a frontal.

Frontal equivalence

Consequences of this construction

Result

The frontal codimensilon of γ is equal to

$$\operatorname{codim}_{\mathscr{A}_e} \gamma - \operatorname{mult} \gamma + 1$$

.

Result

Any corank 1 frontal hypersurface with isolated instability admits a stable frontal unfolding.

Double point space of a frontal surface

Double point space of a frontal surface

Double points split into two branches

If $f(x, y) = (x, p(x, y), q(x, y))$, the **double point space** of f is given by

$$D^2(f) = \left\{ (x, y, y') \in \mathbb{C}^3 \colon \frac{p(x, y) - p(x, y')}{y - y'} = \frac{q(x, y) - q(x, y')}{y - y'} = 0 \right\}$$

If $\pi(x, y, y') = (x, y)$, we set the complex space $D(f) = \pi(D^2(f))$ with the complex space structure induced by π.

Double point space of a frontal surface

The six $\mathscr{F}$-stable frontal surfaces

Result (cf. Marar and Tari, 1996)

If $p_y | q_y$, the generating function for $D(f)$ has the form $\lambda(x, y) = p_y^2(x, y)\tau(x, y)$.

Corollary

Let $\lambda \in \mathscr{O}_2$ be the generating function for $D(f)$, and $\mu = q_y/p_y$:

1. if λ/p_y is regular, f is either a cuspidal edge or a curve of transverse double points;
2. if $V(p_y, \mu_y) = \{0\}$, f is $\mathscr{F}$-finite if and only if λ/p_y has an isolated singularity at 0.

Double point space of a frontal surface

Frontal disentanglement

A smooth family (f_t) is an $\mathscr{F}$-**stabilisation** of a frontal $f\colon (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ if the 1-parameter unfolding $F = (f_t, t)$ is frontal and f_t is frontal stable for $t \neq 0$.

Lemma-Definition

If f has an isolated instability, then it admits an $\mathscr{F}$-stabilisation (f_t) and we call $\Delta_{\mathscr{F}}(f) = \operatorname{Im} f_t$ ($t \neq 0$) the **frontal disentanglement** of f.

Lê Dũng (1987), Siersma (1991)

If f has an isolated instability, $\Delta_{\mathscr{F}}(f)$ has the homotopy type of a wedge of spheres.

Double point space of a frontal surface

Frontal Marar-Mond formulas

Theorem (frontal Marar-Mond formulas)

Given a corank 1 frontal $f\colon (\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ with isolated instability,

$$\mu(f(C),0) = 2S + \mu(C,0);$$
$$2\mu(f(D_+),0) = 2K + 2T + \mu(D_+,0) - W - S + 1$$

- C: Cuspidal edge curve
- S: Swallowtails
- K: Cuspidal double points
- D_+: Transverse double point curve
- T: Triple points
- W: Folded Whitney umbrellas

Double point space of a frontal surface

Counting the invariants

Given a corank 1 frontal $f\colon (\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ with isolated instability,

$$\dim_{\mathbb{C}} \frac{\mathscr{O}_2}{(p_y, p_{yy})} = S; \qquad \dim_{\mathbb{C}} \frac{\mathscr{O}_2}{(p_y, \tau)} = 2S + K + W;$$

$$\dim_{\mathbb{C}} \frac{\mathscr{O}_3}{(p_y, \alpha, \alpha')} = 2S + K; \qquad \dim_{\mathbb{C}} \frac{\mathscr{O}_3}{\mathscr{F}_2(f)} = T + S + K.$$

These identities have been compiled into a Singular library, `frontals.lib`, which can be found at `cuspidalcoffee.github.io`.

Double point space of a frontal surface

Frontal Milnor number

Proposition

Given an analytic plane curve $\gamma\colon (\mathbb{C},0) \to (\mathbb{C}^2,0)$ with $\kappa = |\Sigma(\gamma)| < \infty$,

$$\mu_{\mathscr{F}}(\gamma) = \mu_I(\gamma) - \kappa$$

Theorem

Given a corank 1 frontal $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ with isolated instability,

$$\mu_{\mathscr{F}}(f) = \mu(f(D_+),0) - S - W + T + 1$$

Double point space of a frontal surface
Mond's conjecture

Recall that Mond defines the **image Milnor number** μ_I of $f\colon (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ as the number of spheres in a disentanglement.

Mond's conjecture

If f is $\mathscr{A}$-finite, $\mu_I(f) \geq \operatorname{codim}_{\mathscr{A}_e}(f)$, with equality if and only if f is quasihomogeneous.

Proposed frontal conjecture

Let $f\colon (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ be an $\mathscr{F}$-finite frontal map. Then $\mu_{\mathscr{F}}(f) \geq \operatorname{codim}_{\mathscr{F}_e}(f)$, with equality if and only if f is quasihomogeneous.

Muito obrigado!

Alguma pregunta?

Referencias I