Deformations of frontals: a downstairs theory of Legendrian equivalence

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Abstract

Frontals hypersurfaces are traditionally described as the image of a Legendrian map with singularities, and classified using diffeomorphisms that preserve the Legendrian fibres in the total space. A frontal can also be described intrinsically as a $\mathscr{C}^\infty$ or analytic hypersurface admitting a global tangent hyperplane that varies smoothly relative to the base point. In this joint work with J.J. Nuño-Ballesteros and R. Oset Sinha, we develop a intrinsic classification theory of frontals and showcase some results parallel to the theory of smooth deformations.

Legendrian maps and equivalence

A **contact manifold** is a manifold W^{2n+1} equipped with an $\alpha \in \Omega^1(W)$ such that $\alpha \wedge (d\alpha)^n \neq 0$. Given $w \in W$, we can find a diffeomorphism $\Phi\colon (W,w) \to (PT^*\mathbb{K}^{n+1}, \Phi(w))$ such that

$$\alpha = \Phi^*\sigma; \quad \sigma = \omega_1\, dp^1 + \cdots + \omega_{n+1}\, dp^{n+1}$$

where $\Phi(w) = (p, [\omega])$ ([**Arnold**]).

An **integral map** is a smooth $f\colon N^n \to (W^{2n+1}, \alpha)$ such that $f^*\alpha = 0$. A **Legendrian map** is a pair (f,π), where f is integral and $\pi\colon (W^{2n+1},\alpha) \to Z^{n+1}$ is a submersion such that

$$\ker d\pi_w \subseteq \ker \alpha_w \text{ for all } w \in W.$$

A hypersurface germ $(X,0) \subseteq (Z,0)$ is **frontal** if $X = (\pi \circ f)(N,S)$ for some Legendrian map germ (f,π). We say $(X,0)$ is a **front** if f is immersive.
Two maps $(f,\pi), (f',\pi)$ are equivalent if there exist diffeomorphisms ϕ, ψ, Ψ such that

$$\Psi \circ f = f' \circ \phi; \quad \pi \circ \Psi = \psi \circ \pi'; \quad \alpha = \Psi^*\alpha'$$

An integral deformation of (f,π) is a family (f_u) of integral maps such that $f_0 = f$. We say (f,π) is stable if every integral deformation (f_u) is trivial.

Frontal maps and the Nash lift

A smooth $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ is **frontal** if there exists a nowhere-vanishing $\nu \in \Omega^1(\mathbb{K}^{n+1}, 0)$ such that

$$f^*\nu = 0 \iff \nu(df \circ \xi) = 0 \text{ for all } \xi \in \theta_n$$

A hypersurface $(X,0) \subseteq (\mathbb{K}^{n+1},0)$ is frontal if $(X,0) = f(\mathbb{K}^n, S)$ for some f frontal. Equivalently, $(X,0)$ is frontal if it admits a globally defined tangent hyperplane T_xX that depends smoothly on $x \in (X,0)$.

If f is frontal and $\Sigma(f)$ is nowhere dense, there is a uniquely determined ν verifying the conditions above. We then set $\pi\colon PT^*\mathbb{K}^{n+1} \to \mathbb{K}^{n+1}$ as the canonical projection and define the **Nash lift** of f as $(\tilde{f}, \pi)$, where $\tilde{f} = (f, [\nu])$. We shall call the corank of $\tilde{f}$ the **lift corank** of f.

In corank ≤ 1, we can always take coordinates in the source and target such that

$$f(x,y) = (x, p(x,y), q(x,y)); \quad q_y = \mu p_y \tag{1}$$

for some $p, q, \mu \in \mathscr{O}_n$.

Frontal equivalence and unfoldings

It follows from a result by Arnol'd [**Arnold**] that if f, g are $\mathscr{A}$-equivalent map germs and f is frontal, g is frontal.

An unfolding $F\colon (\mathbb{K}^n \times \mathbb{K}^d, S \times \{0\}) \to (\mathbb{K}^{n+1} \times \mathbb{K}^d, 0)$ of f is frontal if it is frontal as a map-germ. If every frontal unfolding is $\mathscr{A}$-trivial, f is **stable as a frontal** or $\mathscr{F}$-stable.

Result.
Let f be a proper frontal (i.e. $\Sigma(f)$ is nowhere dense). Given a frontal unfolding $F = (u, f_u)$ of f, the family of map germs $\tilde{f}_u$ is an integral deformation of $\tilde{f}$. Conversely, given an integral deformation $\tilde{f}_u$ of f, $F = (u, (\pi \circ f_u))$ is a frontal unfolding of f.

We say f is $\mathscr{F}$-finite if $\operatorname{codim}_{\mathscr{F}_e} f < \infty$, where

$$\mathscr{F}(f) = \left\{ \left.\frac{df_t}{dt}\right|_{t=0} : (f_t, t) \text{ frontal}, f_0 = f \right\}; \quad \operatorname{codim}_{\mathscr{F}_e} f = \dim_{\mathbb{K}} \frac{\mathscr{F}(f)}{T\mathscr{A}_e f}.$$

Result.
A frontal f of lift corank at most 1 is $\mathscr{F}$-stable if and only if it has frontal codimension 0.

A Mather-Gaffney-type criterion for frontals

Result.
Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be given as in (1). If $\operatorname{codim} V(p_y, \mu_y) > 1$, f is $\mathscr{F}$-finite if and only if there is a finite representative of f that is locally $\mathscr{F}$-stable outside S.

Result.
Let $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ be a frontal multigerm with branches $f_1, \dots, f_r$. If f has lift corank 1, then f is $\mathscr{F}$-stable if and only if its branches are $\mathscr{F}$-stable and the spaces

$$\tau(f_i) = \omega f_i^{-1}[tf_i(\theta_n) + (f_i^*\mathfrak{m}_{n+1})\mathscr{F}(f_i)]_0$$

meet in general position, where $_0$ denotes evaluation at the origin.

Result.
A generic frontal hypersurface in $(\mathbb{C}^3, 0)$ contains at most cuspidal edges and transversal intersections near the origin. The double point space of $D(f)$ is then given by the zero locus of the function $p_y^2\tau$, where τ gives the source curve of transversal double points.

The method of frontal reductions

Result.
Let γ be an analytic plane curve with miniversal unfolding Γ. There exists a map h such that the pullback $h^*\Gamma$ is a miniversal frontal unfolding of γ. We denote this pullback by $\Gamma_{\mathscr{F}}$, and call it the frontal reduction of Γ.

Result.
Given an analytic plane curve $\gamma\colon (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$,

$$\operatorname{codim}_{\mathscr{F}} \gamma = \operatorname{codim}_{\mathscr{A}} \gamma - \operatorname{mult}(\gamma) + 1.$$

In particular, if $\mathbb{K} = \mathbb{C}$ and the image of γ is the zero locus of $g \in \mathscr{O}_2$,

$$\operatorname{codim}_{\mathscr{F}} \gamma = \tau(g) - \frac{1}{2}\mu(g) - \operatorname{ord}(g) + 1,$$

where τ, μ denote the Tjurina and Milnor numbers.

Every corank 1 $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ can be seen as an unfolding of a plane curve γ (the *generic slice* of f). The method of frontal reductions allows to construct a stable unfolding of f.

Vanishing homology of a corank 1 frontal

Let (t, f_t) be a 1-parameter stable frontal unfolding of $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$. It follows from a result by Lê that the image of f_t for $t \neq 0$ has the homology type of a wedge of spheres, the number of which is independent on the choice of unfolding. We denote this image by $\Delta_{\mathscr{F}}(f)$, and call it the **frontal disentanglement** of f. We call the number of spheres the **frontal Milnor number** $\mu_{\mathscr{F}}$ of f.

Result.
If f has isolated $\mathscr{F}$-instability, we have

$$\mu_{\mathscr{F}}(f) = \mu_I(f) - \kappa \qquad (n=1); \qquad \mu_{\mathscr{F}}(f) = \mu(f(D(f)), 0) - S - W + T + 1 \qquad (n=2)$$

First expression is a consequence of a result by Giménez Conejero and Nuño-Ballesteros [**GCNB**].

Result.
Let $C = V(p_y)$ be the cuspidal edge curve of f, and $D_+ = V(\tau)$ be the proper double point curve of f. If f has isolated frontal instability and $V(p_y, \mu_y) = \{0\}$,

$$\mu(f(C(f)), 0) = 2S + \mu(C(f), 0); \qquad 2\mu(f(D(f)), 0) = 2K + 2T + \mu(D(f), 0) - W - S + 1$$

Conjecture.
Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathscr{F}$-finite frontal map. Then $\mu_{\mathscr{F}}(f) \geq \operatorname{codim}_{\mathscr{F}_e}(f)$, with equality if and only if f is quasihomogeneous.

Conclusions

The theory of deformations of frontals allowed us to gain some insight into the topology of frontal maps of corank at most 1. In particular, we found that some of the classic results from the theory of smooth deformations still apply to the smaller space of corank ≤ 1 frontals, albeit with some modifications. We believe the next logical step is to find conditions under which these statements hold in corank 2, such as arbitrary frontal surfaces in $\mathbb{K}^3$.

References

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