Singularities of frontals

A new approach in the classification of frontal germs

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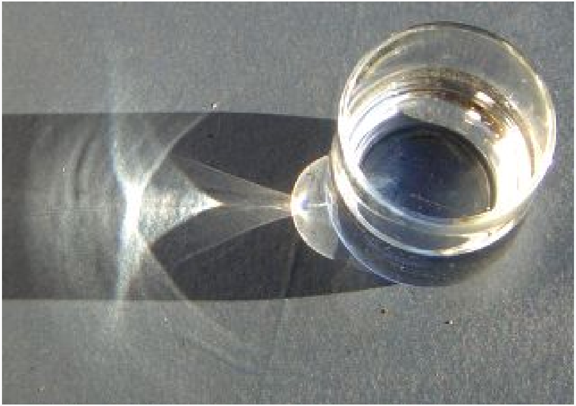
Outline

Introduction

Caustics and wave fronts

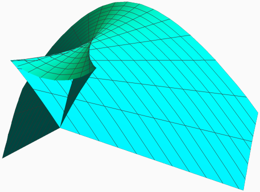


(Heiner Otterstedt, 2006)

From wave fronts to frontals

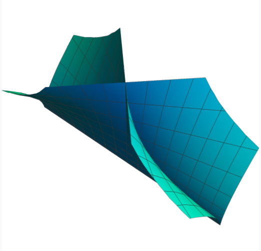
Wave front:

$$\begin{array}{ccc} L & \hookrightarrow & PT^*\mathbb{K}^{n+1} \\ & {\scriptstyle f}\searrow & \downarrow \\ & & \mathbb{K}^{n+1} \end{array}$$



Frontal:

$$\begin{array}{ccc} \mathbb{K}^n & \xrightarrow{F} & PT^*\mathbb{K}^{n+1} \\ & {\scriptstyle f}\searrow & \downarrow \\ & & \mathbb{K}^{n+1} \end{array}$$



Equidistant hypersurfaces

Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be an immersion, $Z = f(U)$ and $\xi\colon U \to \mathbb{R}^{n+1}$ a unit vector field along f. The **equidistant hypersurfaces** to Z are defined as the hypersurfaces Z_t given by

$$f_t(x) = f(x) + t\xi(x); \qquad\qquad x \in U,\ t \in \mathbb{R}$$

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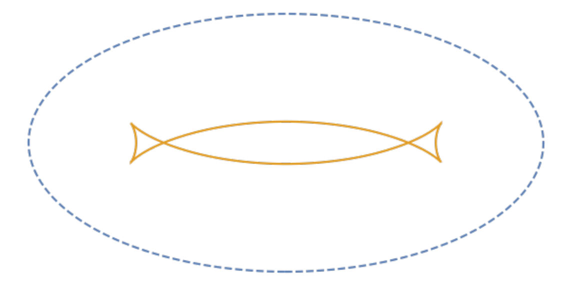

Legendre equivalence

Legendrian fibrations

Let $h\colon PT^*\mathbb{K}^{n+1} \to \mathbb{K}^{n+1}$ be the canonical projection onto the base space. A submersion $\pi\colon PT^*\mathbb{K}^{n+1} \to \mathbb{K}^{n+1}$ is a **Legendrian fibration** if, given $p = (q, [\omega]) \in PT^*\mathbb{K}^{n+1}$,

$$\ker d\pi_p \subseteq \ker(\omega \circ dh_p)$$

An **integral mapping** is a smooth map $F\colon U \subset \mathbb{K}^n \to PT^*\mathbb{K}^{n+1}$ such that, for all $x \in U$

$$\operatorname{Im} dF_x \subseteq \ker(\omega \circ dh_{F(x)})$$

where $F(x) = (q, [\omega])$.

Legendre equivalence

Two pairs $(F, \pi), (G, \pi')$ are **Legendrian equivalent** if we can find diffeomorphisms ϕ, ψ and a contactomorphism Ψ such that the squares in the following diagram commute:

$$\begin{array}{ccccc}
(\mathbb{K}^n, 0) & \xrightarrow{F} & PT^*\mathbb{K}^{n+1} & \xrightarrow{\pi} & \mathbb{K}^{n+1} \\
\phi\big\downarrow & & \big\downarrow\Psi & & \big\downarrow\psi \\
(\mathbb{K}^n, 0) & \xrightarrow{G} & PT^*\mathbb{K}^{n+1} & \xrightarrow{\pi'} & \mathbb{K}^{n+1}
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Theorem

In the above diagram, Ψ is locally determined by π, π' and ψ.

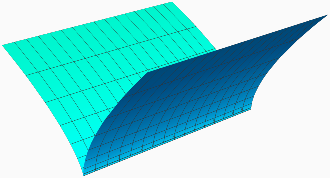
Stability of integral mappings

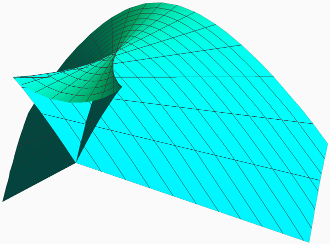
An **integral deformation** of $F\colon (\mathbb{K}^n, 0) \to PT^*\mathbb{K}^{n+1}$ is a family of integral maps (F_u) that depends smoothly on $u \in (\mathbb{K}^d, 0)$ such that $F_0 = F$.

We say a pair (F, π) is **Legendre stable** if, for each integral deformation (F_u), we can find (ϕ_u), (ψ_u) and (Ψ_u) such that

$$\Psi_u \circ F \circ \phi^{-1} = F_u; \qquad \pi \circ \Psi_u = \psi_u \circ \pi$$

Mind the second square





Not Legendrian equivalent,

Mind the second square

$$L\colon (x, y, 0, 0, 0) \xrightarrow{\pi} (x, y^3 + 3xy, y^4 + 2xy^2)$$

$$L\colon (x, y, 0, 0, 0) \xrightarrow{\pi'} (x, y^2, y^3)$$

Not Legendrian equivalent, *but they come from the same immersion*.

Open Whitney umbrellas

G. Ishikawa performed an extensive analysis of the notion of Legendrian stability in his 2005 article. Some of his findings include:

- Integral deformations can be constructed using a differential form $\tilde{\alpha}$, called **Nash lift**.

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- Integral deformations can be constructed using a differential form $\tilde{\alpha}$, called **Nash lift**.
- A pair (π, F) is Legendrian stable if and only if F is an **open Whitney umbrella** and the algebra

$$Q(F) = \frac{F^*\mathscr{O}_{PT^*\mathbb{K}^{n+1}}}{(\pi \circ F)^*\mathfrak{m}_{n+1}F^*\mathscr{O}_{PT^*\mathbb{K}^{n+1}}}$$

is generated by a certain family of functions.

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  is generated by a certain family of functions.
- Open Whitney umbrellas are classified by their **type**.

Frontal equivalence

Frontal germs

A germ $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ is **frontal** if there exist a representative $f\colon U \to V$ such that $f(U)$ has a well-defined tangent space $T_{f(u)}f(U)$ for all $u \in U$.

Taking coordinates $(x_1, \ldots, x_n, y)$ on $\mathbb{K}^{n+1}$, f is a frontal germ if and only if

$$d(y \circ f) = \sum_{j=1}^{n} p_j \, d(x_j \circ f)$$

for some $p_1, \ldots, p_n\colon (\mathbb{K}^n, 0) \to \mathbb{K}$.

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for some $p_1, \dots, p_n\colon (\mathbb{K}^n, 0) \to \mathbb{K}$. A **Nash lifting** of f is an integral $\bar{f}\colon (\mathbb{K}^n, 0) \to PT^*\mathbb{K}^{n+1} \equiv \mathbb{K}^{n+1} \times \mathbb{K}^n$ given by

$$\bar{f}(u) = (f(u); p_1(u), \dots, p_n(u))$$

Examples

- Every analytic plane curve is frontal.
- Let $f\colon (\mathbb{K}^2,0) \to (\mathbb{K}^3,0)$ be the germ

$$f(x,y) = (x, y^2, xy)$$

If f is frontal, there must exist a unit vector field ξ such that

$$\langle f_x(x,y), \xi\rangle = \langle f_y(x,y), \xi\rangle = 0$$

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These equations are equivalent to

$$\xi_1+y\xi_3=0; \qquad 2y\xi_2+x\xi_3=0$$

but no unit vector field verifies these equations.

Frontal maps are preserved under $\mathscr{A}$-equivalence

Proposition

Let $f, g\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$. If f is frontal and g is $\mathscr{A}$-equivalent to f, then g is a frontal.

$$\begin{array}{ccc} (\mathbb{K}^n, S) & \xrightarrow{f} & (\mathbb{K}^{n+1}, 0) \\ \phi\big\downarrow & & \big\downarrow\psi \\ (\mathbb{K}^n, S) & \xrightarrow{g} & (\mathbb{K}^{n+1}, 0) \end{array}$$

Frontal stability

A **frontal unfolding** of f is a germ

$$F_d\colon (\mathbb{K}^n \times \mathbb{K}^d, S \times \{0\}) \to (\mathbb{K}^{n+1} \times \mathbb{K}^d, 0)$$

such that

1. for all $x \in (\mathbb{K}^n, S)$, $F_d(x, 0) = (f(x), 0)$;
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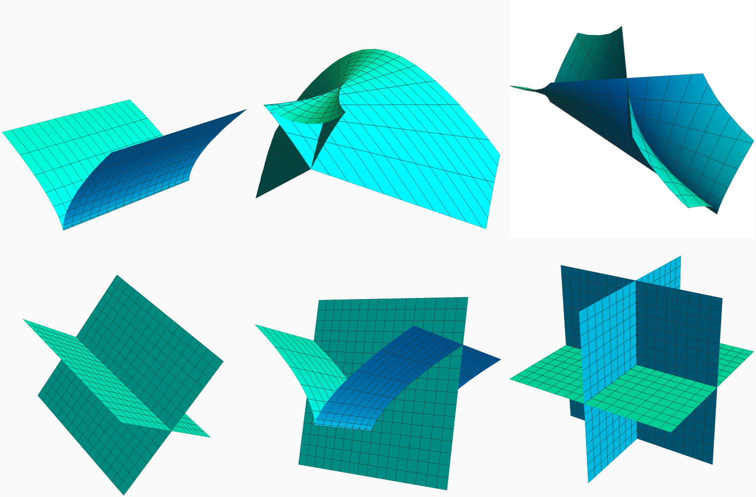
A frontal unfolding is **trivial** if it is $\mathscr{A}$-equivalent to $f \times \mathrm{id}_{(\mathbb{K}^d,0)}$ for some d. If every frontal unfolding of f is trivial, we say f is **stable as a frontal**.

The stable frontal surfaces



Characterizing frontal stability (i)

Definition

Given a frontal f with Nash lifting $\overline{f}$, we set

$$\mathscr{F}(f) = \left\{ \left.\frac{df_s}{ds}\right|_{s=0} : (f_s, s) \text{ frontal} \right\};$$

$$T\mathscr{A}_e f = \left\{ \left.\frac{df_s}{ds}\right|_{s=0} : f_s = \psi_s \circ f \circ \phi_s^{-1} \right\}$$

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We define the **frontal codimension** of f as

$$\operatorname{codim}_{\mathscr{F}_e} f = \dim_{\mathbb{K}} \frac{\mathscr{F}(f)}{T\mathscr{A}_e f}$$

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Proposition

Let f be a corank 1 frontal germ. If f is generically immersive, f is stable as a frontal if and only if it has frontal codimension 0.

Classifying stable frontals by their local algebras

We denote $\mathscr{O}_n$ the ring of smooth germs $(\mathbb{K}^n, 0) \to \mathbb{K}$. Given a mapping $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, $f^*\mathfrak{m} \subseteq \mathscr{O}_n$ denotes the ring generated by the component functions of f. We set

$$Q(\overline{f}) = \frac{\mathscr{O}_n}{\overline{f}^*\mathfrak{m}}; \qquad Q_I(f) = \frac{\overline{f}^*\mathscr{O}_{2n+1}}{(f^*\mathfrak{m})\overline{f}^*\mathscr{O}_{2n+1}}$$

(Ishikawa, 2005)

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(Ishikawa, 2005)

Conjecture

Two stable frontal germs $f, g\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^{n+1}, 0)$ are $\mathscr{A}$-equivalent if and only if $Q(\overline{f}) \cong Q(\overline{g})$ and $Q_I(f) \cong Q_I(g)$.

Characterizing frontal stability (ii)

We define

$$\hat{\tau}(f_i) = \omega f_i^{-1}[tf_i(\theta_n) + (f_i^*\mathfrak{m})\mathscr{F}(f_i)]|_0$$

where $|_0$ denotes evaluation at 0.

Characterizing frontal stability (ii)

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Proposition

Let f be a frontal multi-germ with branches $f_1, \ldots, f_r$. Then f is stable as a frontal if and only if $f_1, \ldots, f_r$ are stable and $\hat{\tau}(f_1), \ldots, \hat{\tau}(f_r)$ meet in general position.

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Conjecture

The space $tf(\theta_n) + (f^*\mathfrak{m})\mathscr{F}(f)$ is the equivalent in frontal equivalence to the $\mathscr{K}$-tangent space in Mather's theory of $\mathscr{A}$-equivalence.

Frontal Mather-Gaffney criterion

Proposition

Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a corank 1 frontal mapping. If f is finite and codim $V(p_y, \lambda_y) > 1$, it has finite frontal codimension if and only if there is a small enough representative $f\colon X \to Y$ such that:

1. $f^{-1}(0) = S$;
2. the restriction $f\colon X \backslash f^{-1}(0) \to Y \backslash \{0\}$ is locally stable as a frontal.

Double point curve of a frontal surface

The double point curve

Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ be a parametrized surface of corank 1. If

$$f(x,y) = (x, p(x,y), q(x,y))$$

the **double point space** of f is given by

$$D^2(f) = \left\{ (x,y,y') \in \mathbb{C}^3 \colon \frac{p(x,y)-p(x,y')}{y-y'} = \frac{q(x,y)-q(x,y')}{y-y'} = 0 \right\}$$

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If $\pi\colon \mathbb{C}^3 \to \mathbb{C}^2$ is the proyection given by $(x,y,y') \mapsto (x,y)$, we shall write $D(f) = \pi(D^2(f))$. The curve $D^2(f)$ induces an algebraic structure on $D(f)$ by taking the Fitting ideals of π.

Double point curve of a frontal surface

Let $f\colon(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ be a frontal germ of corank 1 and λ be the generating function of $D(f)$. If $f(x,y)=(x,p(x,y),q(x,y))$, either $p_y|q_y$ or $q_y|p_y$ (Nuño-Ballesteros, 2015).

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Proposition

1. If $p_y | q_y$, $p_y^2 | \lambda$.
2. If λ / p_y is regular, f is either a cuspidal edge or a curve of transversal double points.

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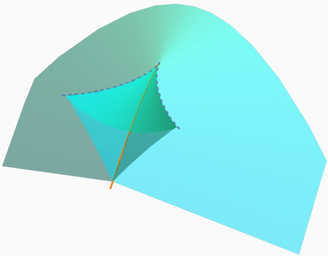
Proposition

1. If $p_y|q_y$, $p_y^2|\lambda$.
2. If λ/p_y is regular, f is either a cuspidal edge or a curve of transversal double points.
3. The germ f is stable as a frontal if and only if λ/p_y has an isolated singularity at 0.

Double point curve and frontal stability



$$f(x,y) = (x, y^3 + 3xy, y^4 + 2xy^2)$$

- Space of double points:

$$\lambda(x,y) = (x+y^2)^2(3x+y^2)$$

- Cuspidal edge set:

$$p_y(x,y) = x + y^2$$

- Double point set:

$$\tau(x,y) = 3x + y^2$$

Open questions

Open questions

- Can we compute the module $\mathscr{F}(f)$ on Singular?
- The corank 1 condition is a limitation imposed by Ishikawa (2005). Does any of these results hold in corank 2?
- In Mather's theory, a germ is finitely $\mathscr{A}$-determined if and only if it has finite $\mathscr{A}$-codimension. Does this hold for frontals?
- Marar-Mond number for frontal surfaces.

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